A crash course on Quantum Information and Quantum Cryptography

Ricardo Faleiro, IT-Aveiro

13/11/24

This project was funded within the DIGITAL-2021-QCI-01 Programme that has received funding from the European Union's Horizon 2020 research and innovation programme under the Grant Agreement No. 101091730.

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Quantum Information Theory

False

Conversation w/ Steve Wiesner, who told me that:

A variation on the Einstein-Rosen-Podolsky Gedankenexperiment can be used to send, through a channel with a nominal capacity of one bit, two bits of information; subject however to the constraint that, ~~the receiver may at his choice read either~~ whichever bit the receiver chooses to read, the other bit is destroyed.

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This paper treats a class of codes made possible by restrictions on measurement related to the uncertainty principal. Two concrete examples and some general results are given.

Conjugate Coding *

Stephen Wiesner

Columbia University, New York, N.Y.
Department of Physics

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Goals	Understanding and predicting physical phenomena (e.g., atomic structures, scattering processes, decays).	Manipulating and processing information, often for tasks like quantum computing, communication, and cryptography

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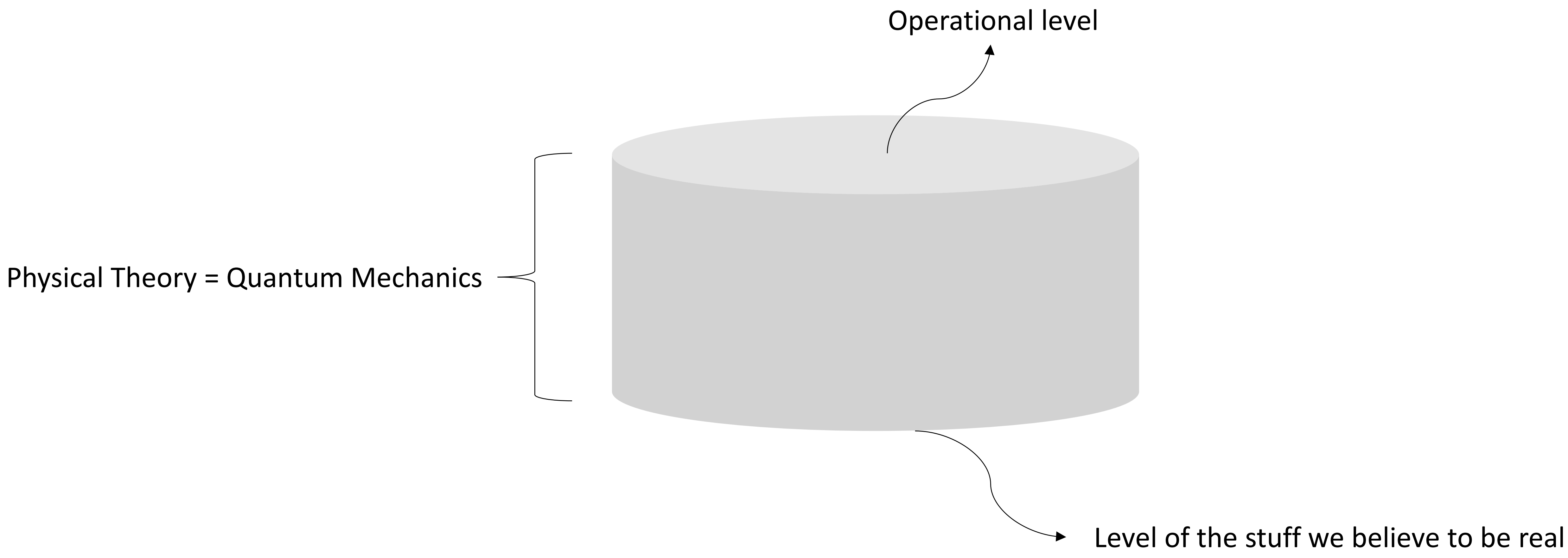
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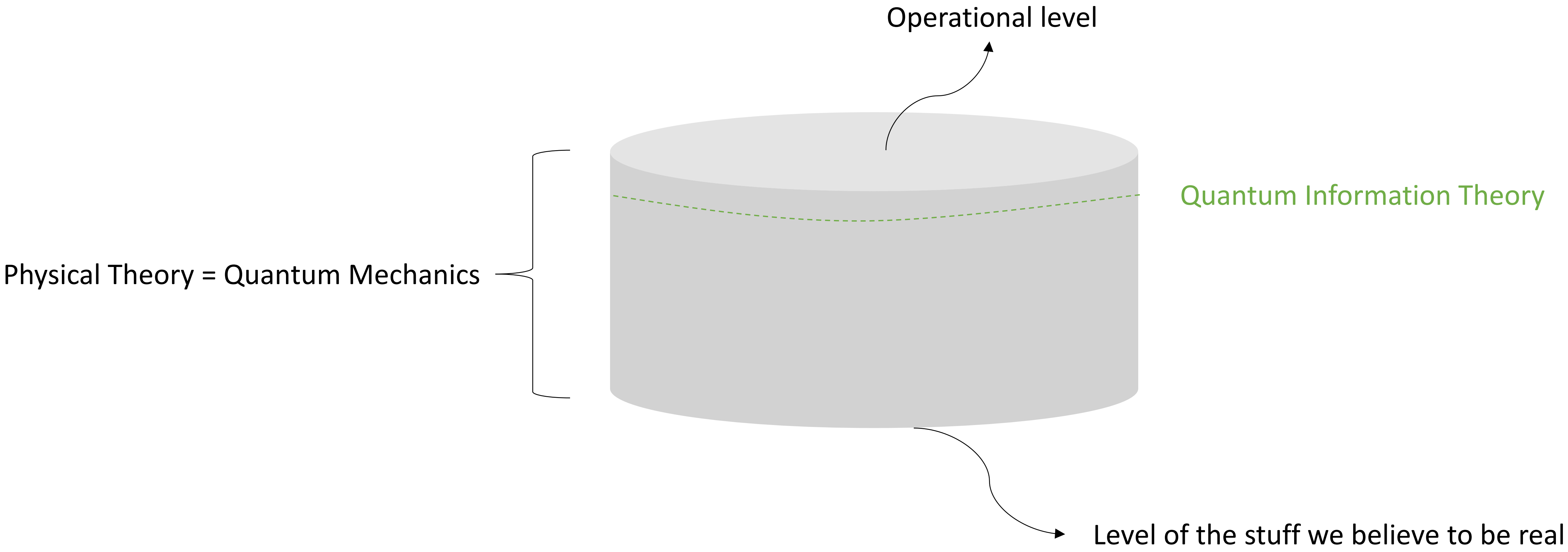
Physical Theory = Quantum Mechanics
Level of the stuff we believe to be real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“But if quantum mechanics isn’t physics in the usual sense — if it’s not about matter, or energy, or waves, or particles — then what is it about? From my perspective, it’s about information and probabilities and observables, and how they relate to each other.” – Scott Aaronson, Quantum Computing Since Democritus

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So what is different? The theory is the same, but taken from a new perspective:

- **QM is a physical theory grounded in a notion of physicalism, that is, it focuses on "real" physical systems and their properties;**

- **QI is an epistemic framework concerned with studying the manipulation and processing of information emergent from the quantum mechanical phenomena**.

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E.g. Typically, one tries to be as agnostic as possible about the contents of the boxes, but sometimes general assumptions can be established. For instance, we may consider preparations that only output states up to a certain dimension or energy.

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- Coherence, superposition;
- Measurement incompatibility;
- Entanglement;
- Nonlocality

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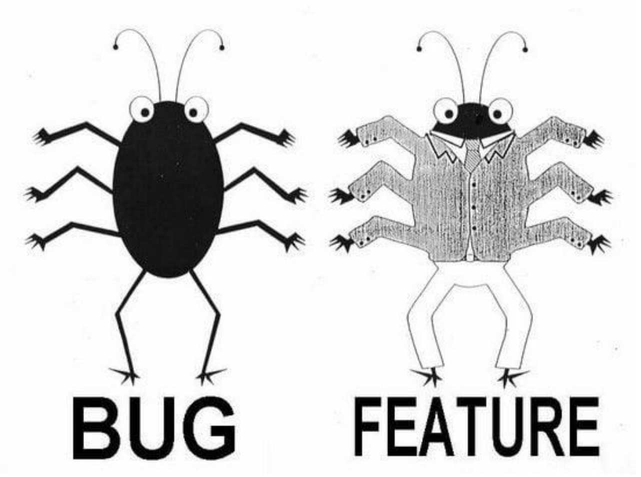

(Deterministic) Preparations

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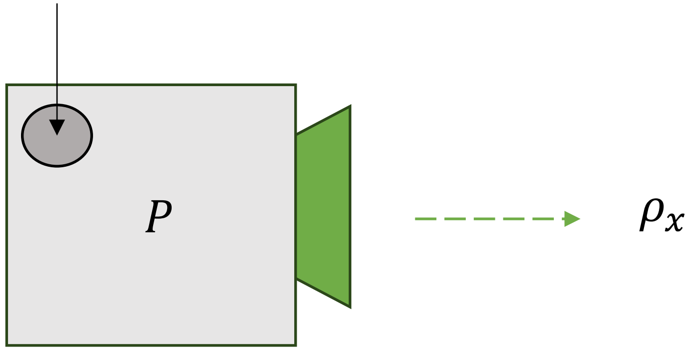

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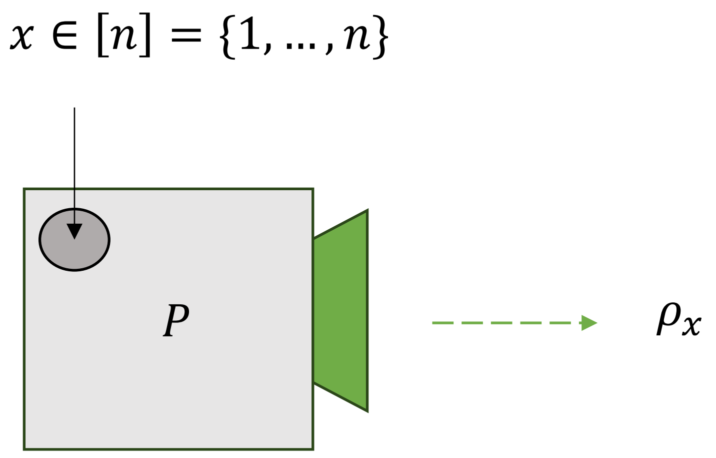

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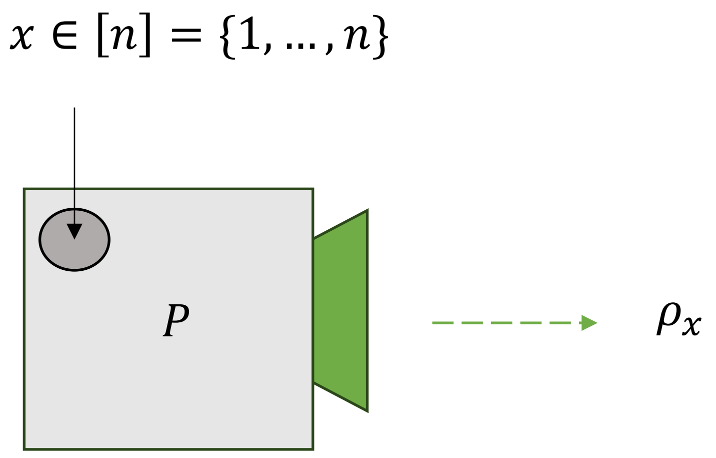

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- $\langle \boldsymbol{H}, \{\rho_x | \, x \in [n]\} \rangle$
  - $\boldsymbol{H}$ = Hilbert space

(For finite dimension, can be assumed to be a complex Euclidean space. A linear (vector) field with the Euclidean metric and usual inner product of vectors in $\mathbb{C}^n$, where the vectors, operators etc, can have over complex numbers as components.)

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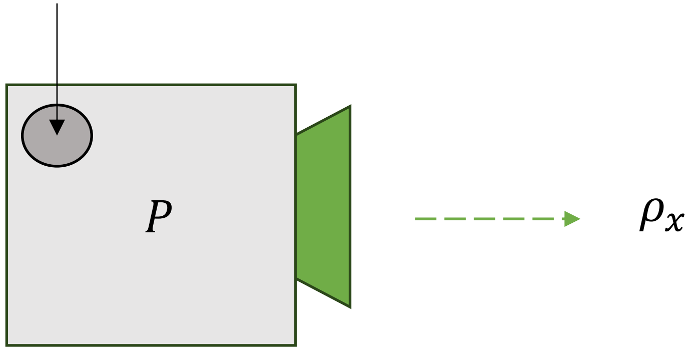

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Each ρ_x is called a ***density operator***. They are, positive semi-definite, trace one, Hermitian operators acting on a Hilbert space $\boldsymbol{H}$.

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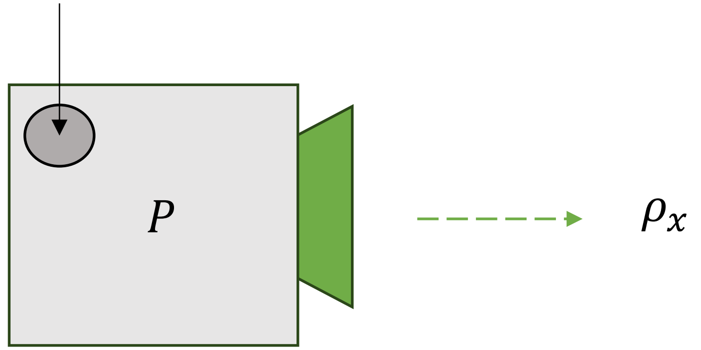

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Hermiticity: $\rho_x = \rho_x^\dagger$

Positive Semi-Definiteness: For any vector $| \psi \rangle \in \boldsymbol{H}$, we have $\langle \psi | \rho | \psi \rangle \geq 0$. Equivalent to say that the spectrum, the set of eigenvalues, is non-negative.

Trace Condition: $Tr(\rho) = 1$

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$\downarrow$

$Span\{|0\rangle, |1\rangle\}$

2D Quantum system = Qubit

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- $\{\rho_1 = |0\rangle\langle 0|, \rho_2 = |1\rangle\langle 1|, \rho_3 = |+\rangle\langle +|, \rho_4 = |-\rangle\langle -|\}$

Where,
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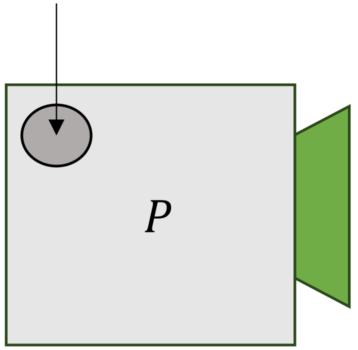

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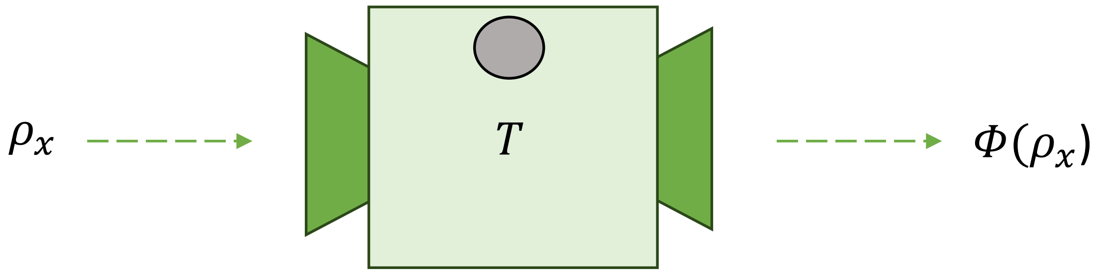

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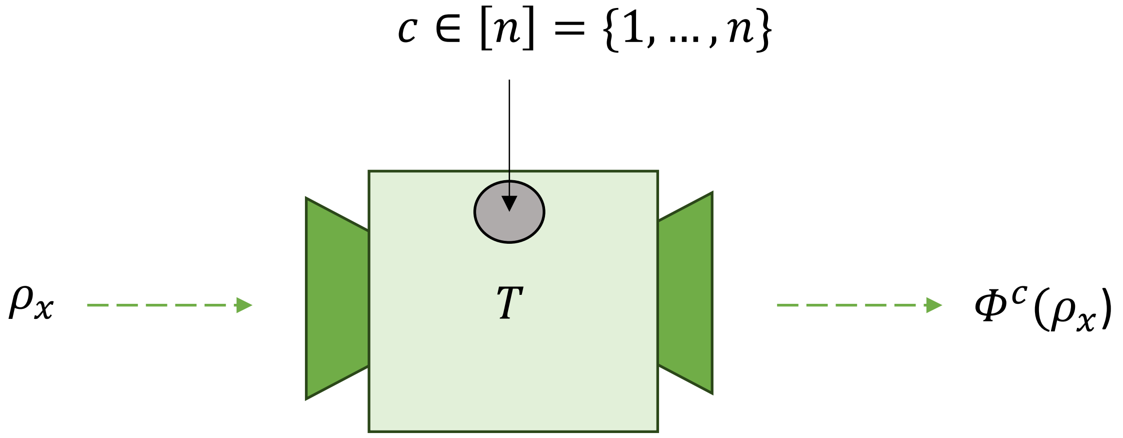

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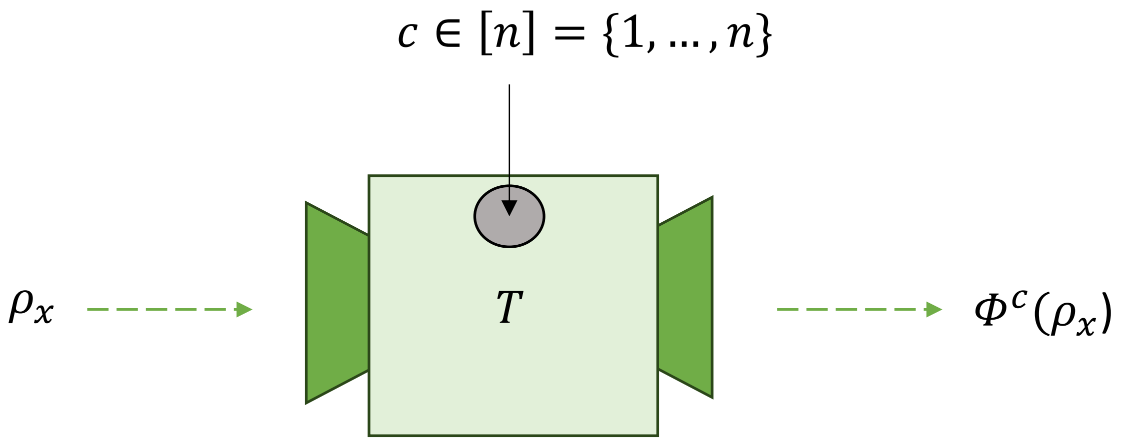

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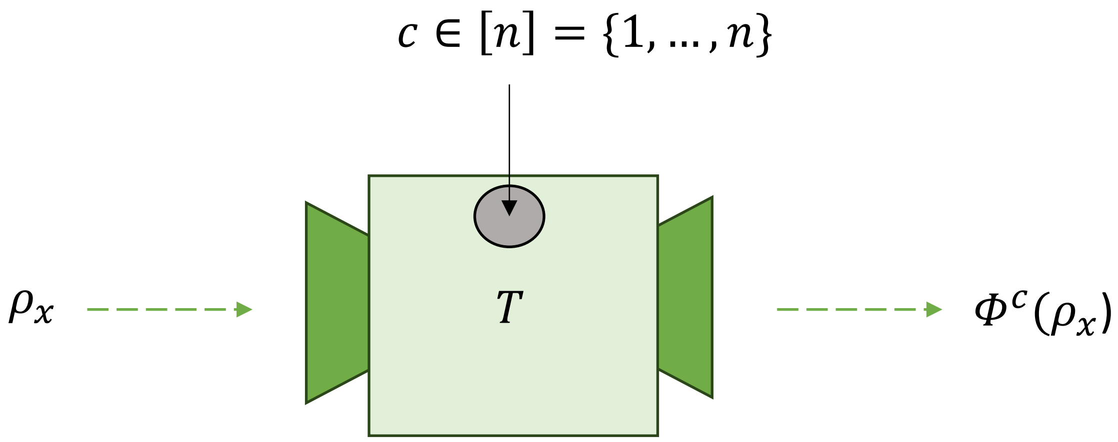

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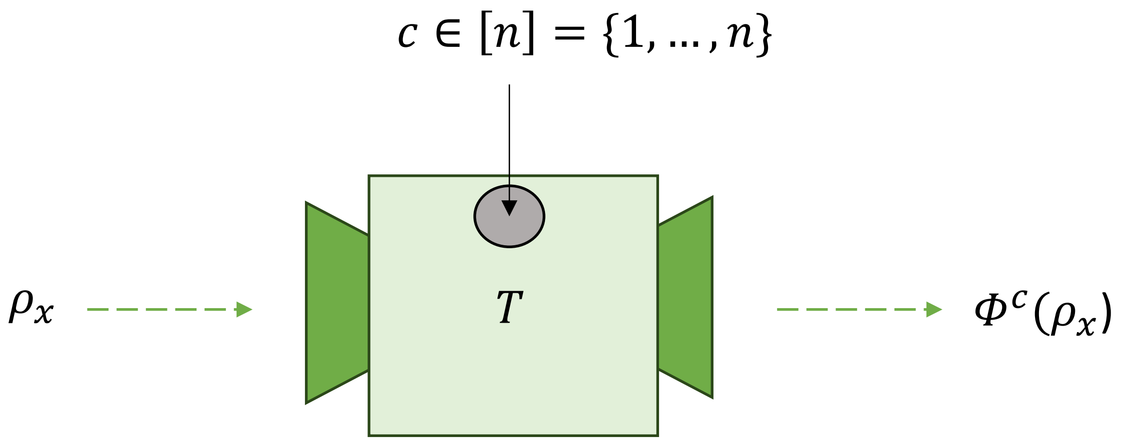

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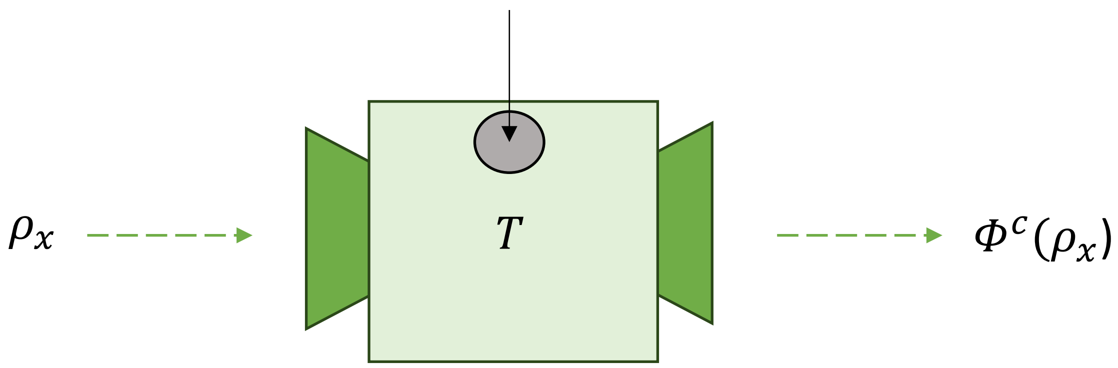

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Complete Positivity: A map Φ is completely positive if, for any state ρ, $\Phi(\rho)$ is positive semi-definite. This ensures that no negative probabilities arise.

Trace Preservation: $Tr\big(\Phi(\rho)\big) = Tr(\rho)$ for all ρ.

To guarantee this property, the Kraus operators must satisfy : $\sum_i K_i^\dagger K_i = I$, where I is the identity operator on the Hilbert space $\boldsymbol{H}$.

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- $\Phi^1(\rho_x) = \rho_x$; $\Phi^2(\rho_x) = H \rho_x H^\dagger$

Special case, where transformations are given by unitaries U,i.e there is only one Kraus operator, the unitary itself, $U^\dagger U = I$

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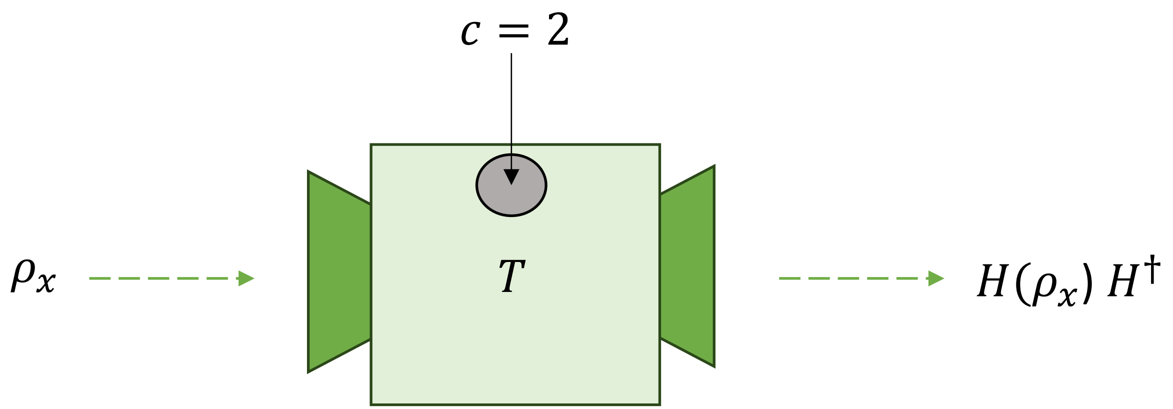

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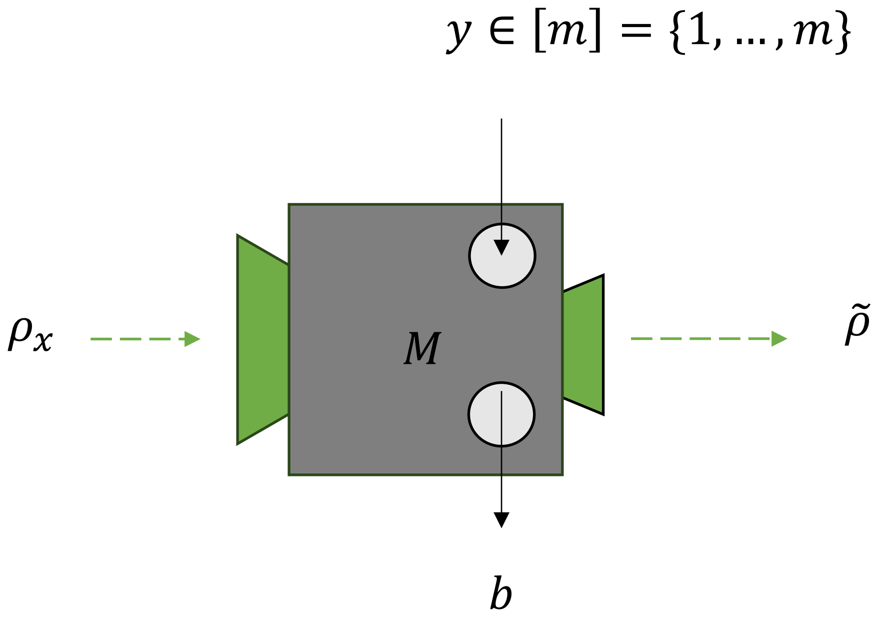

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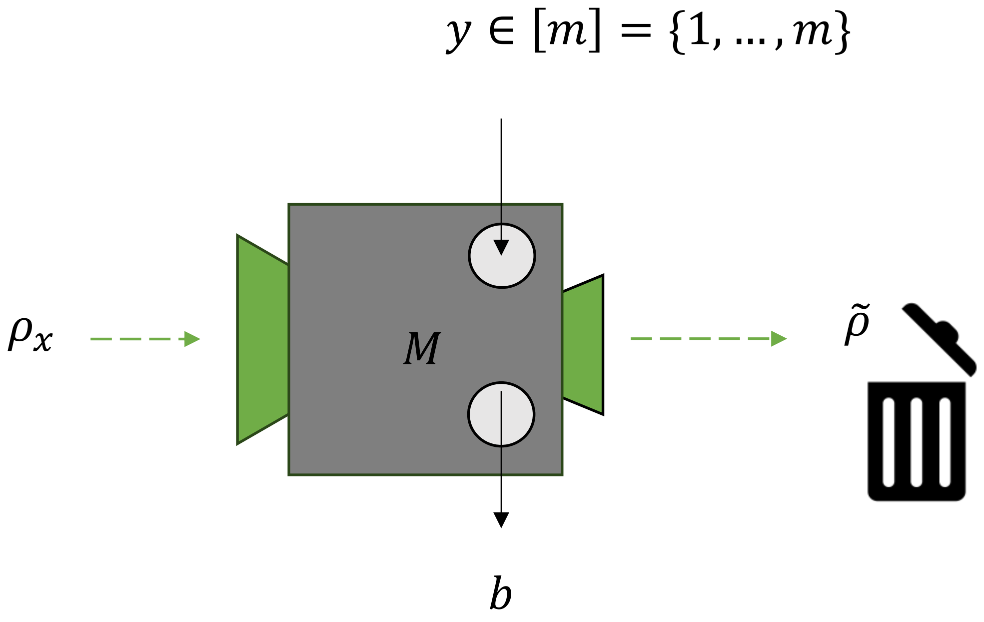

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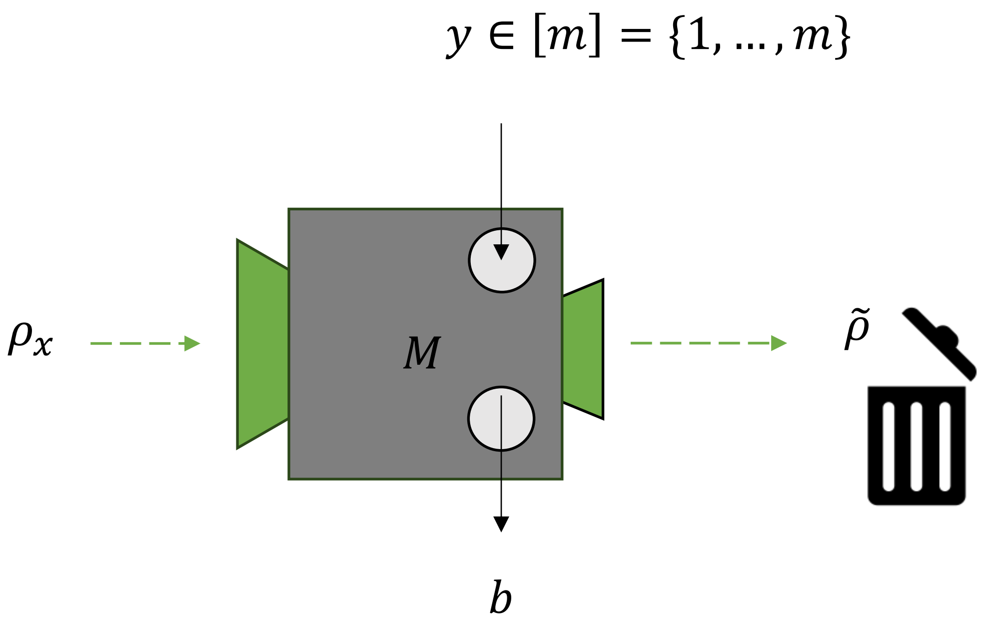

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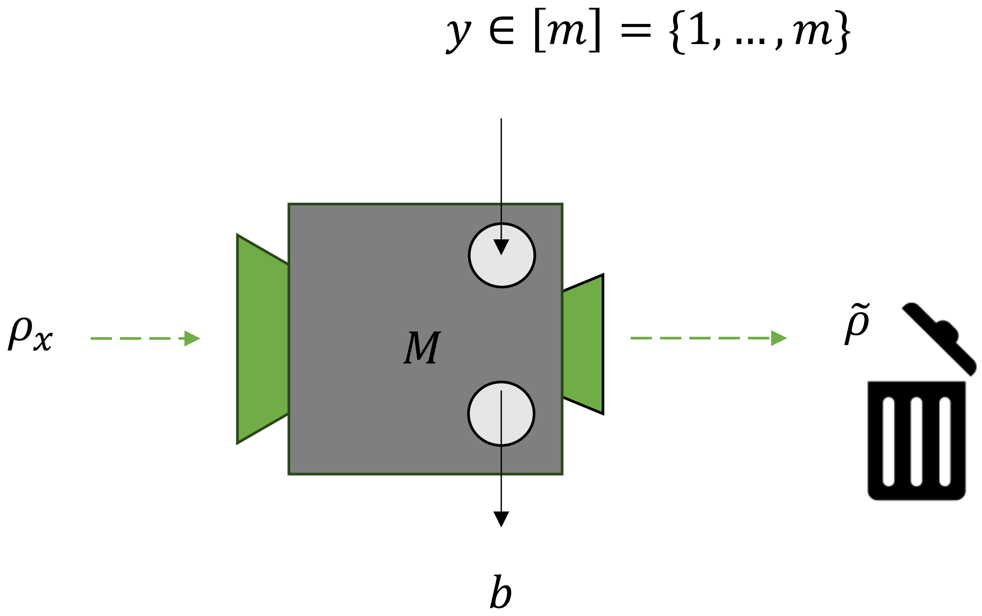

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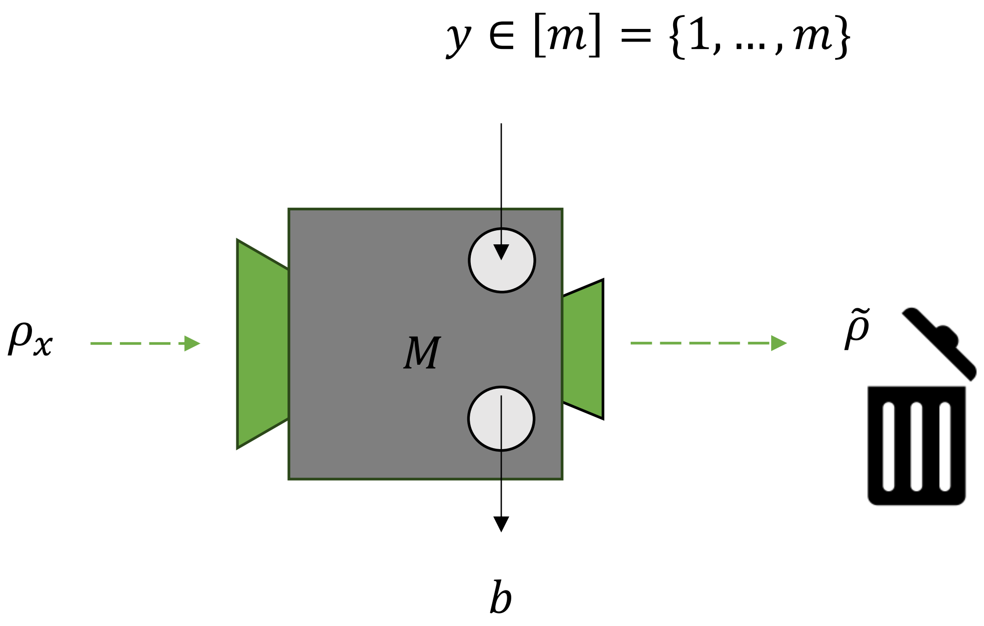

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Sum to identity: For all y, $\sum_b E_{b|y} = I_H$,

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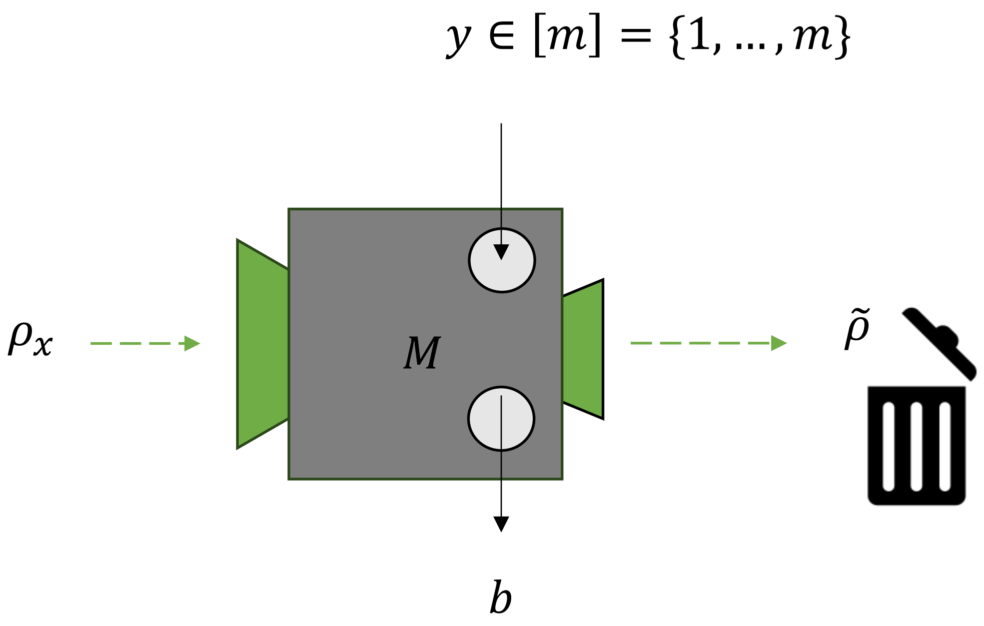

In the lab it might look more like this…

- $\langle \boldsymbol{H}, \{E_{b|y}\}_y \rangle$
  - $\{E_{b|y}\}$ is Positive Operator Valued Measure (POVMs)— a measurement is given by specifying one POVM for each choice of y. POVMs elements are positive semi-definite, Hermitian operators acting on a Hilbert space $\boldsymbol{H}$.

Hermiticity: $E_{b|y} = E_{b|y}{}^{\dagger}$

Positive Semi-Definiteness: For any vector $|\,\psi\rangle \in \boldsymbol{H}$, we have $\langle\psi\,|\,E_{b|y}\,|\,\psi\rangle \geq 0$. Equivalent to say that the spectrum, the set of eigenvalues, is non-negative.

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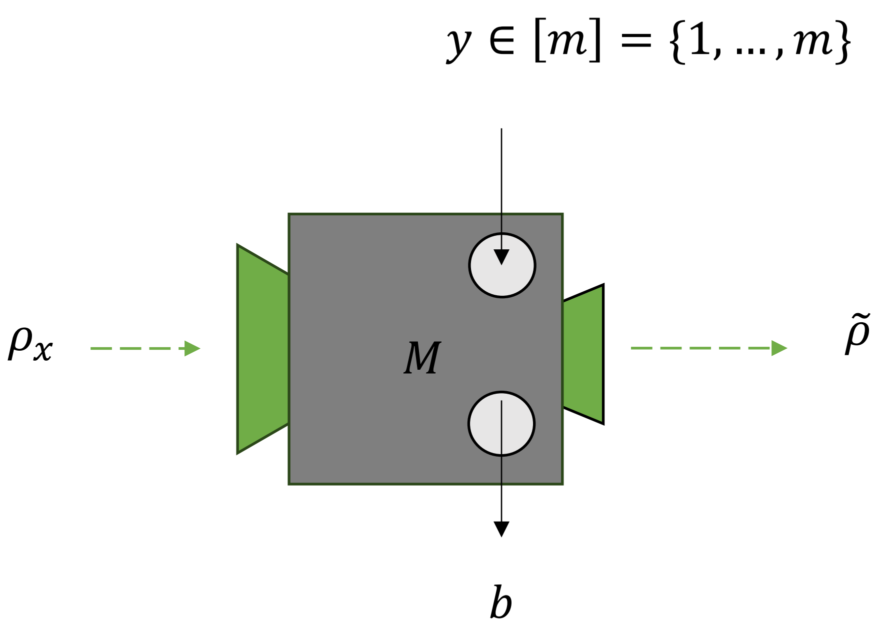

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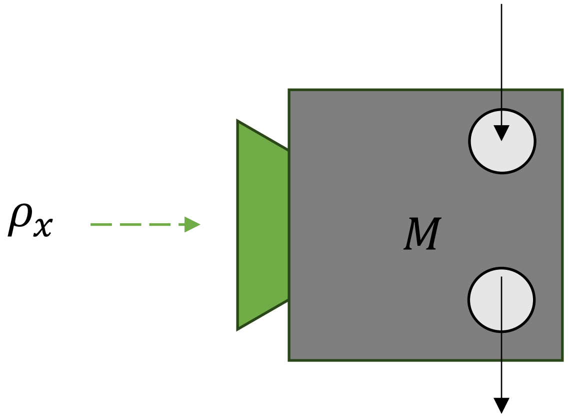
$$E_{b|y}{}^{\dagger} = K_{b|y}K_{b|y}{}^{\dagger}$$

Measurements (destructive)

$y \in [m] = \{1, \dots, m\}$



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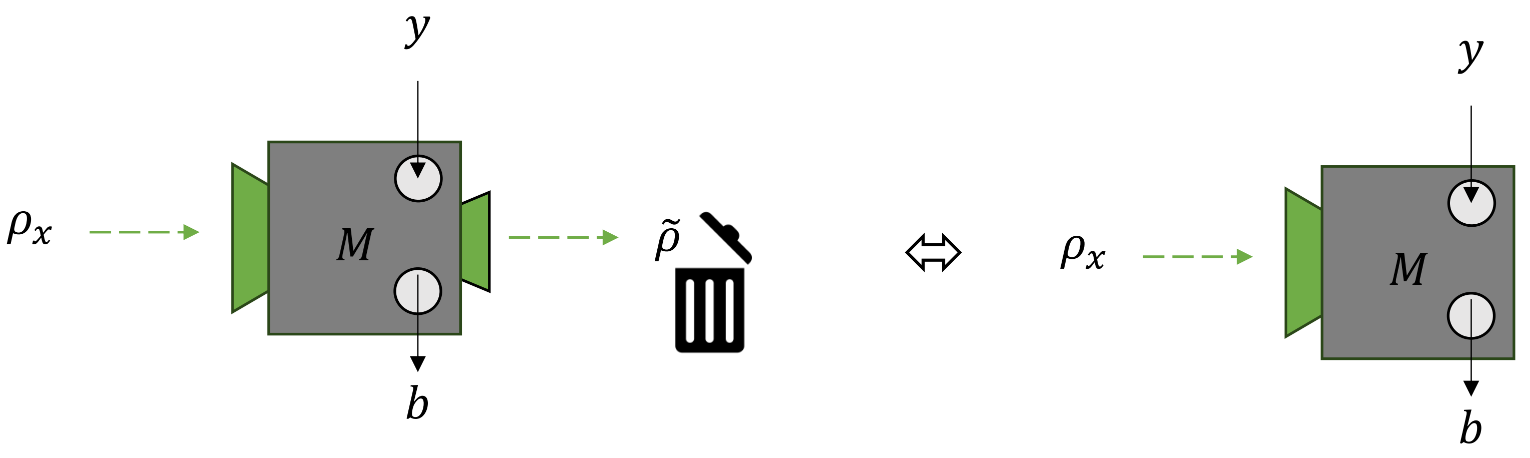
Operationally, to trash the system is equivalent to assume that it did not exist.



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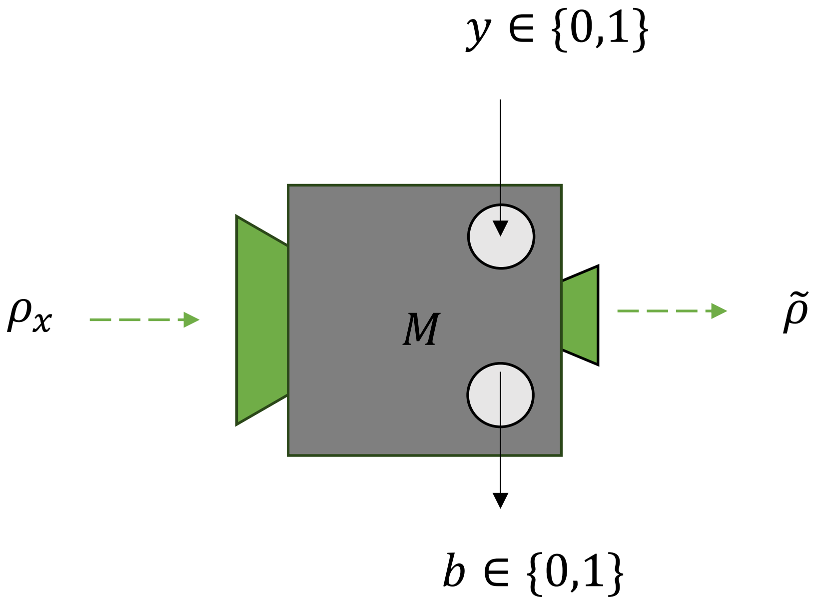

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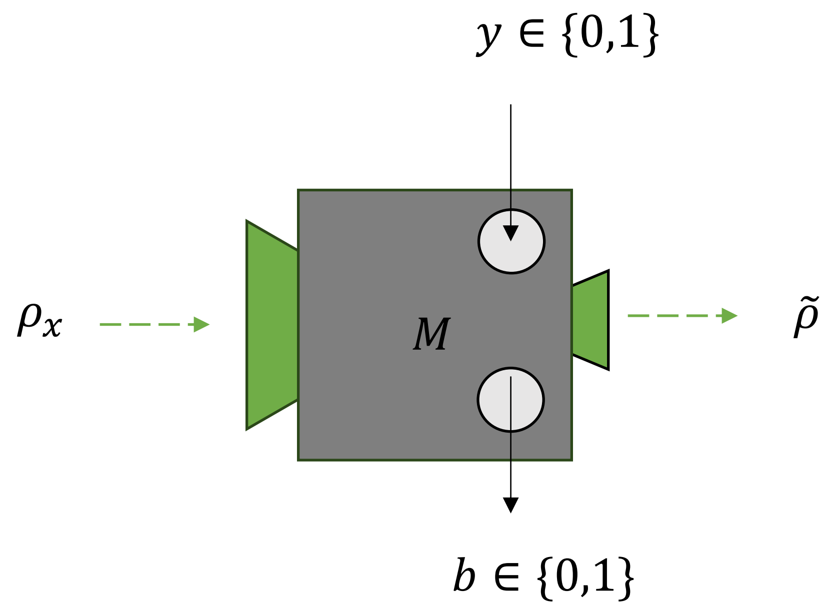

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Calculating the post-measurement state for projective measurements is easier, it is just the state associated with the classical value registered y.

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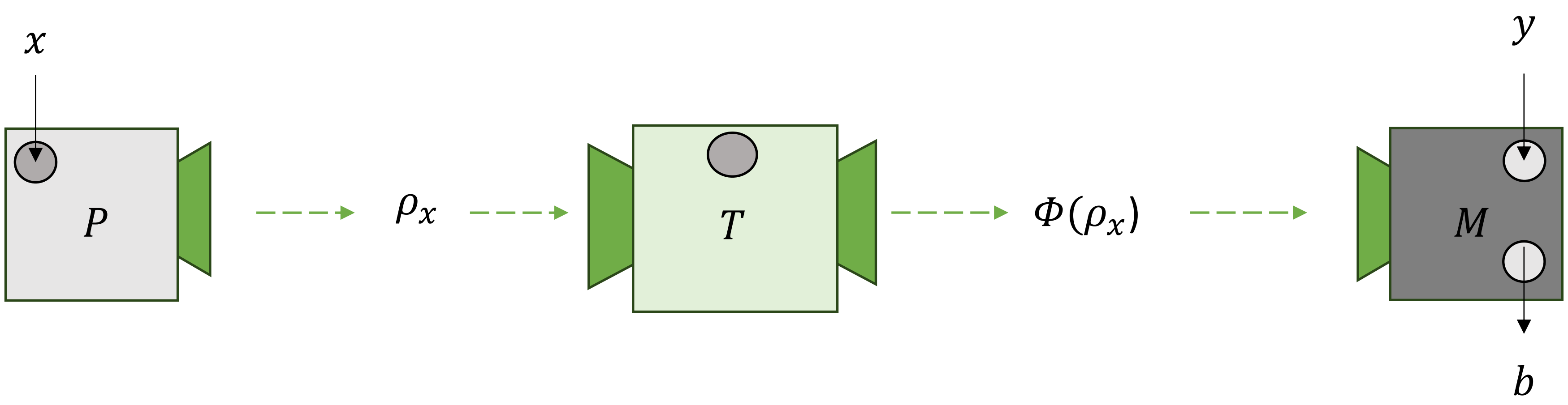

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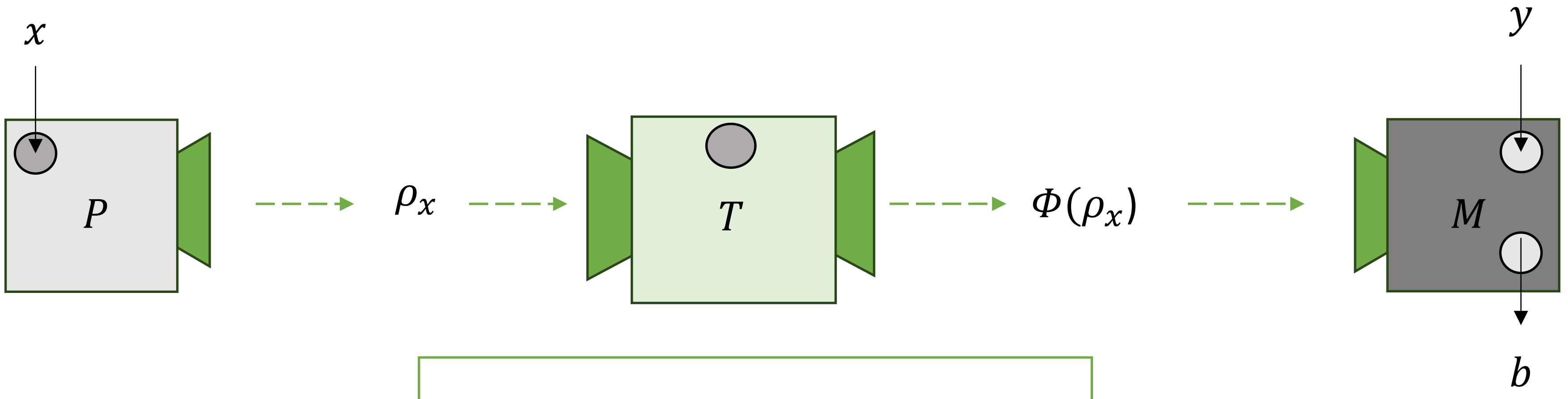

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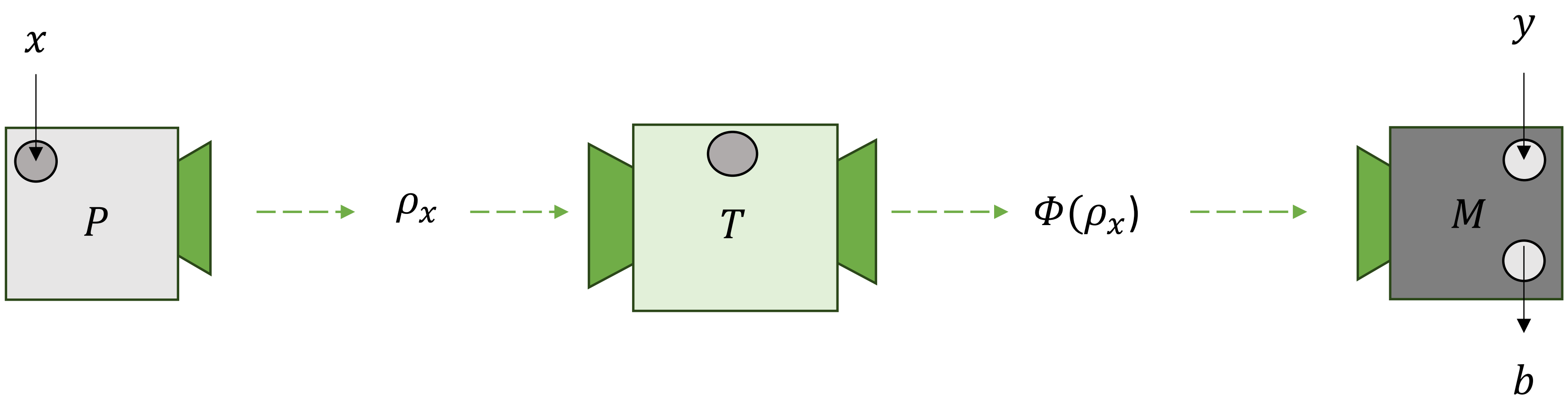
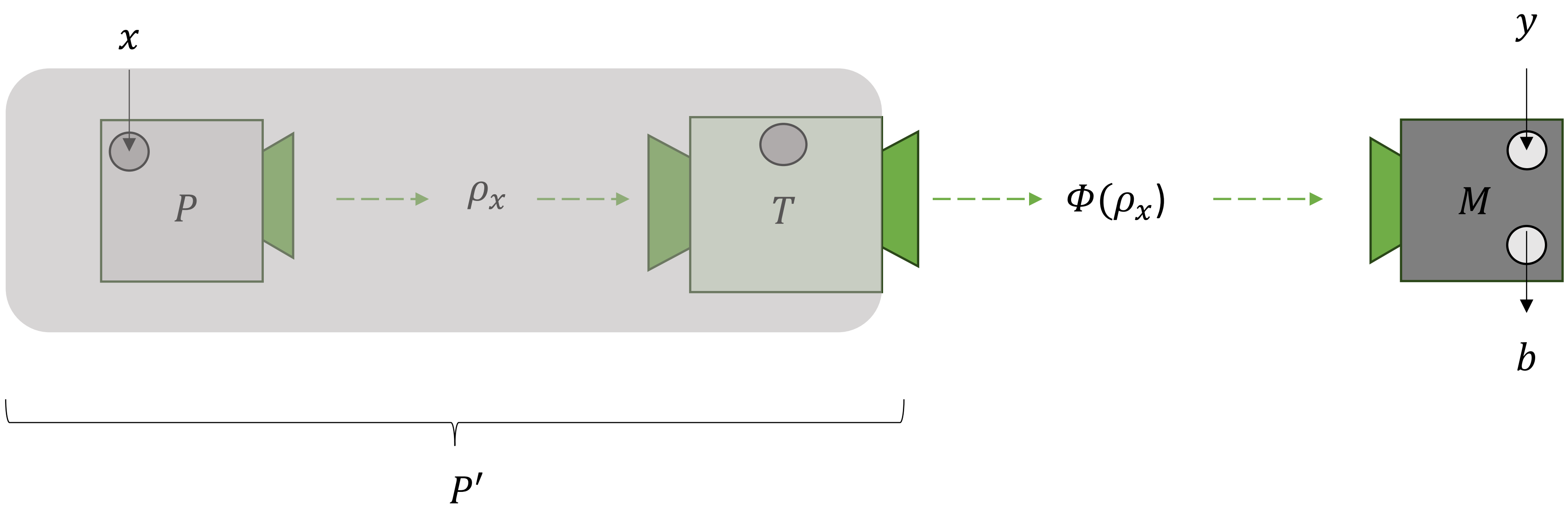
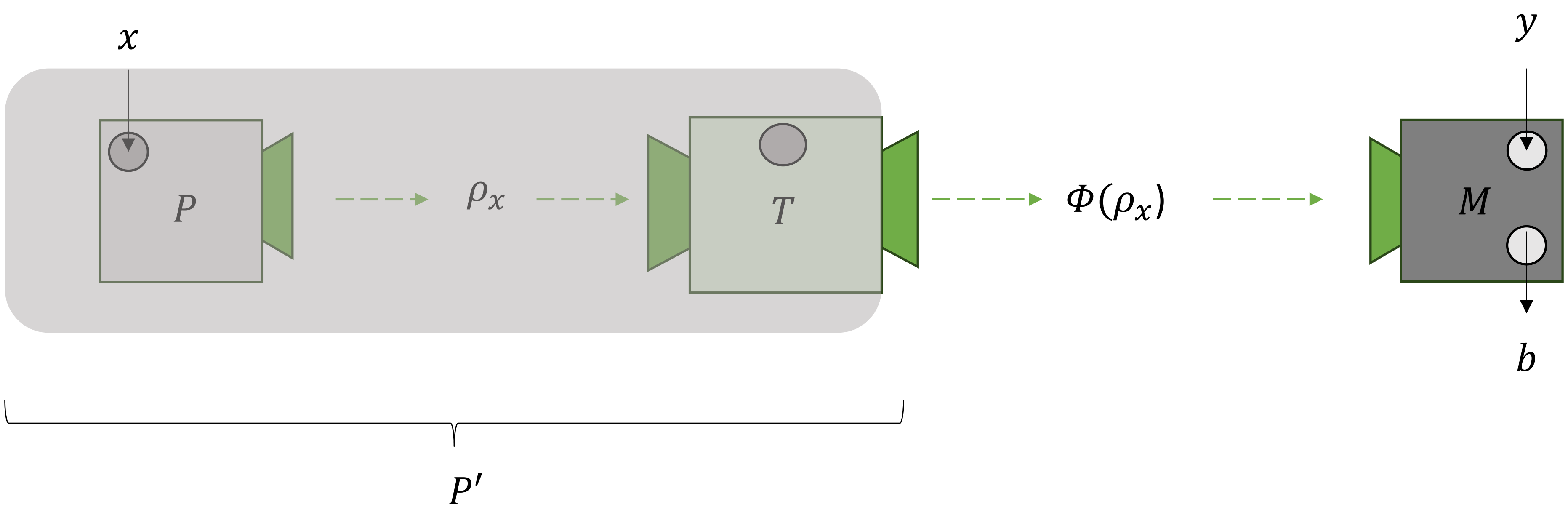
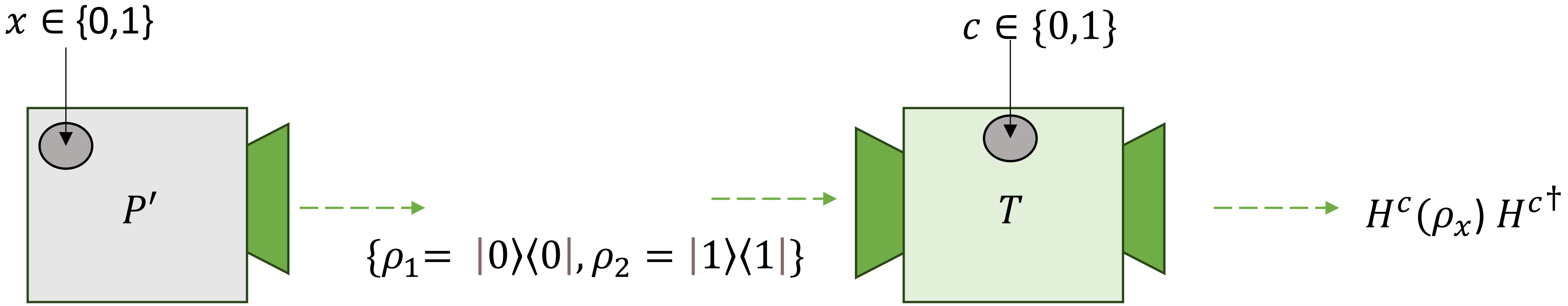


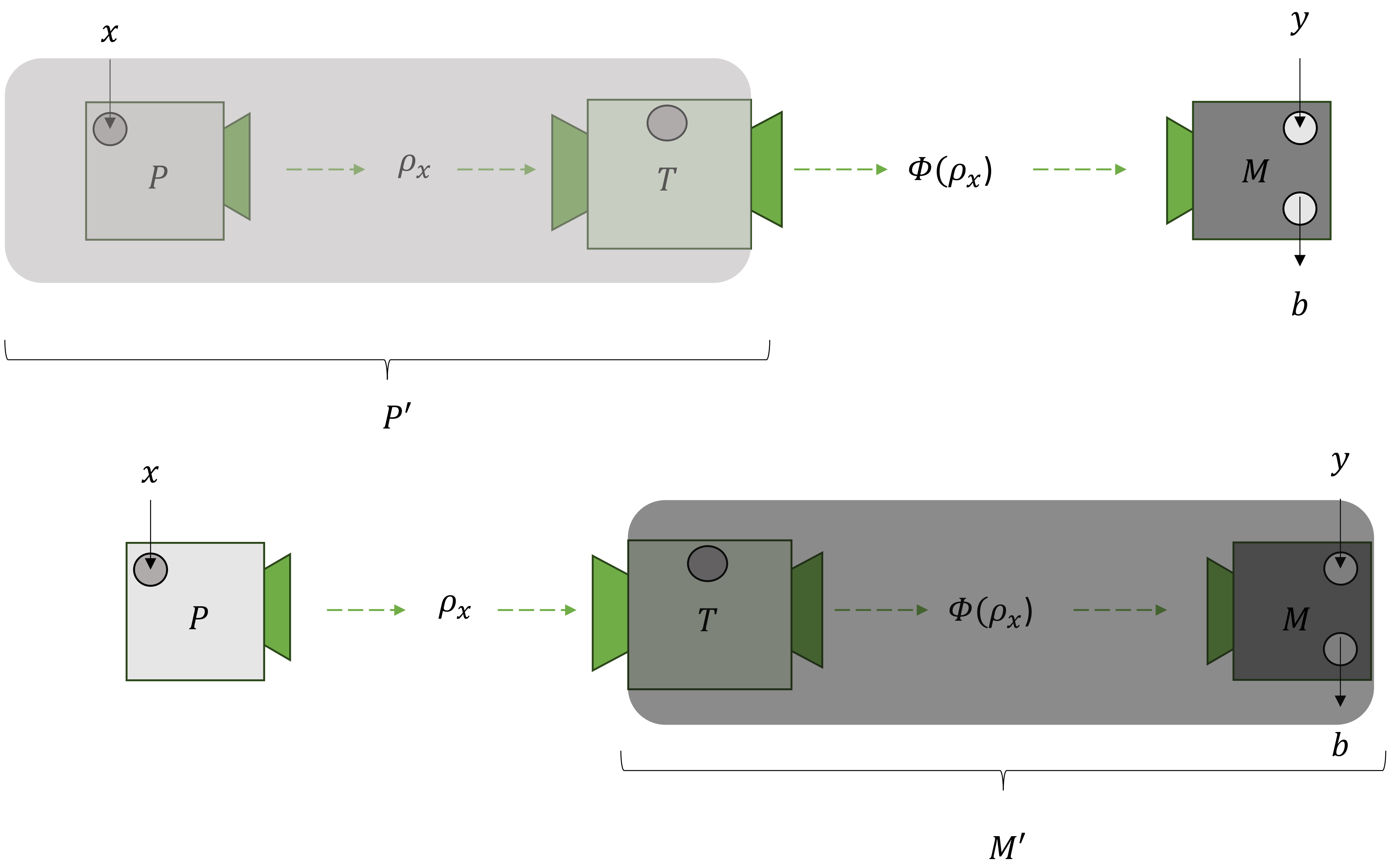
x
P
ρ_x
T
$\Phi(\rho_x)$
y
M
b


x
P
ρ_x
T
Φ(ρ_x)
M
y
b
P′





This is equivalent to the original example of the preparation we saw.


x
P
ρ_x
T
Φ(ρ_x)
y
M
b
P′
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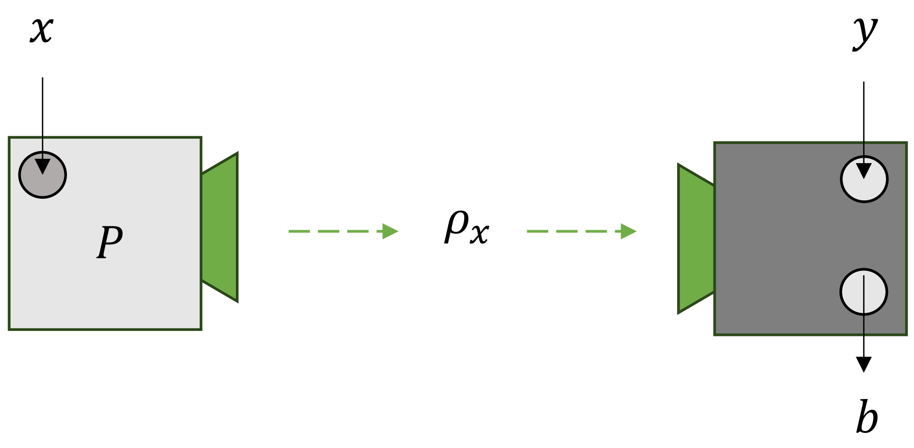
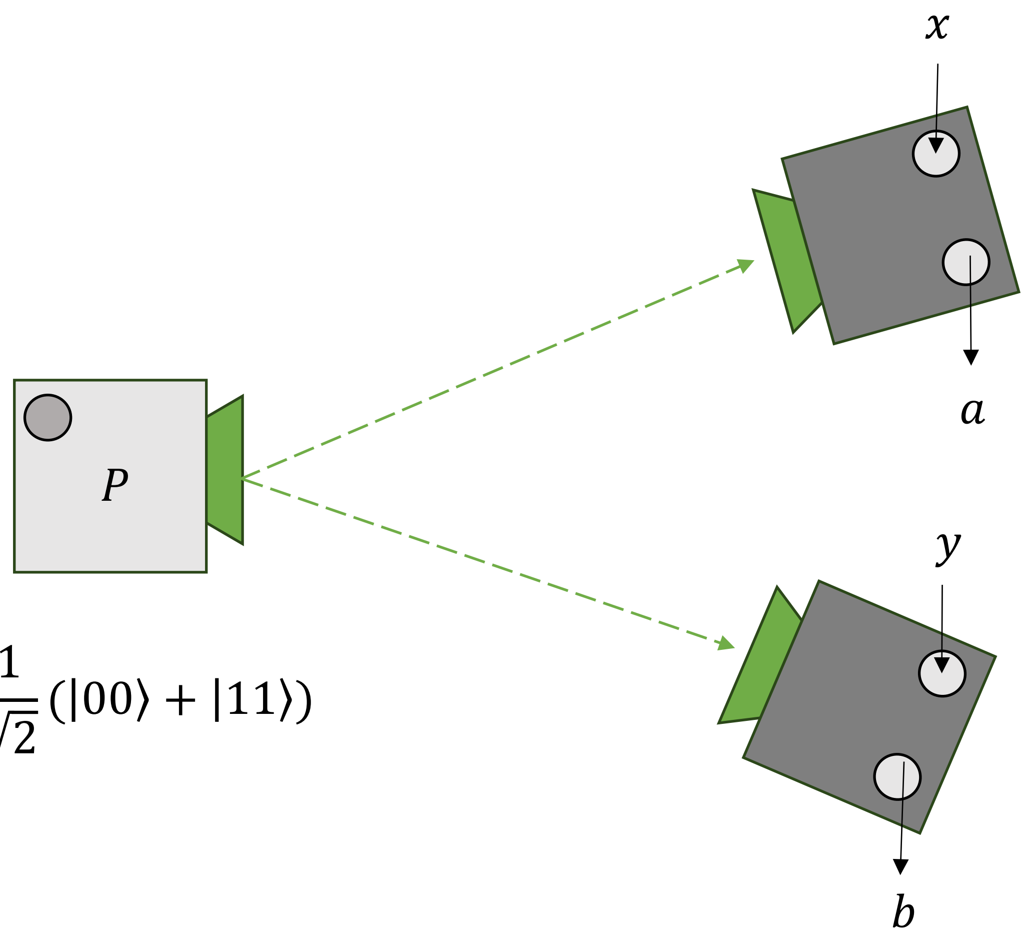
Prepare and Measure scenario



$$p(b|y,x) = Tr(\rho_x E_{b|y})$$

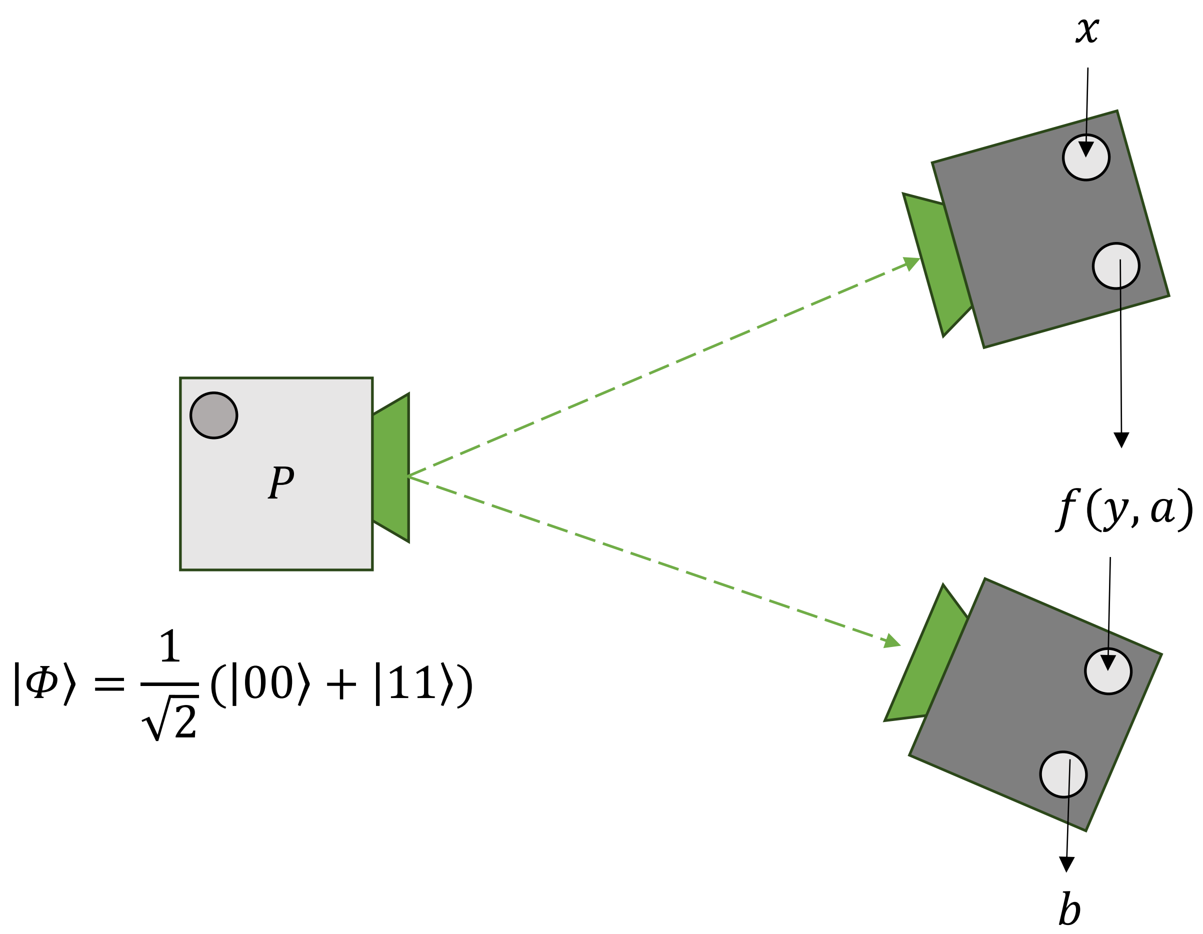
1 Preparation and 2 Measurements ?

1 Preparation and 2 Measurements: Bi-partite Bell (Nonlocality)

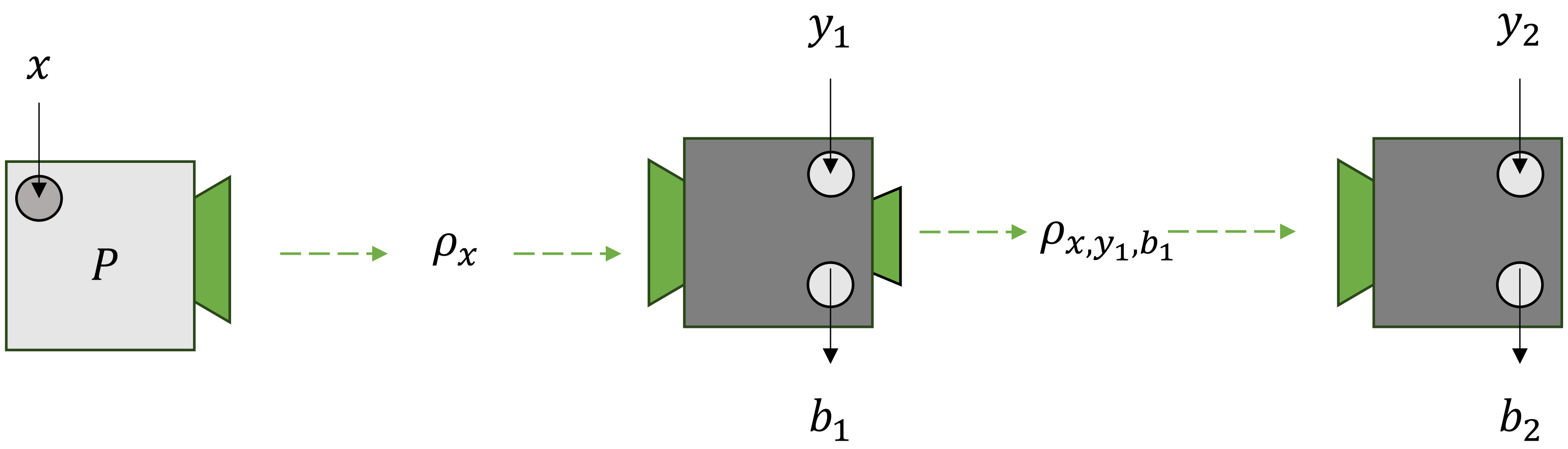


$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

1 Preparation and 2 Measurements: Bi-partite Bell w/ restricted classical communication



1 Preparation and 2 Measurements: Sequential Measurement Scenario with two measurements (Legget-Garg Inequalities, Temporal correlation, KS-Contextuality)



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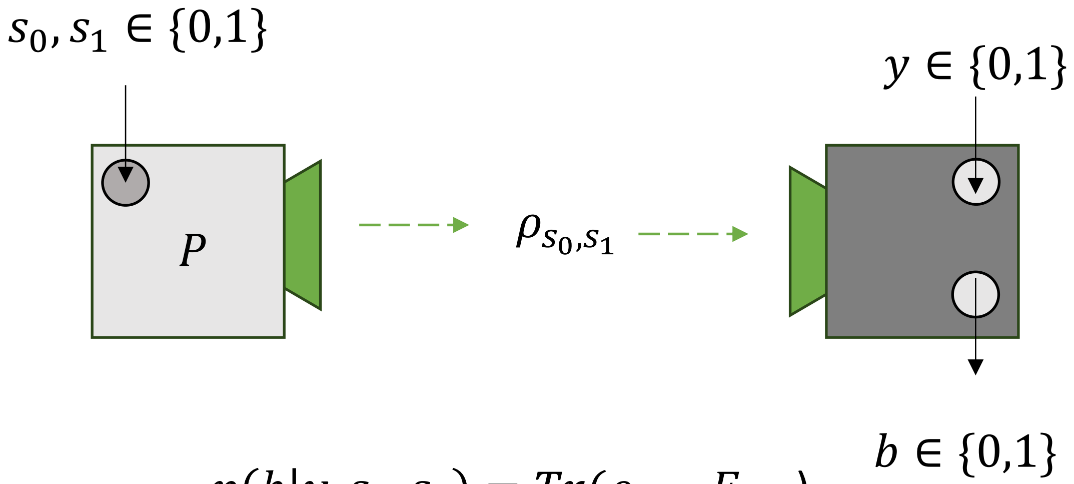

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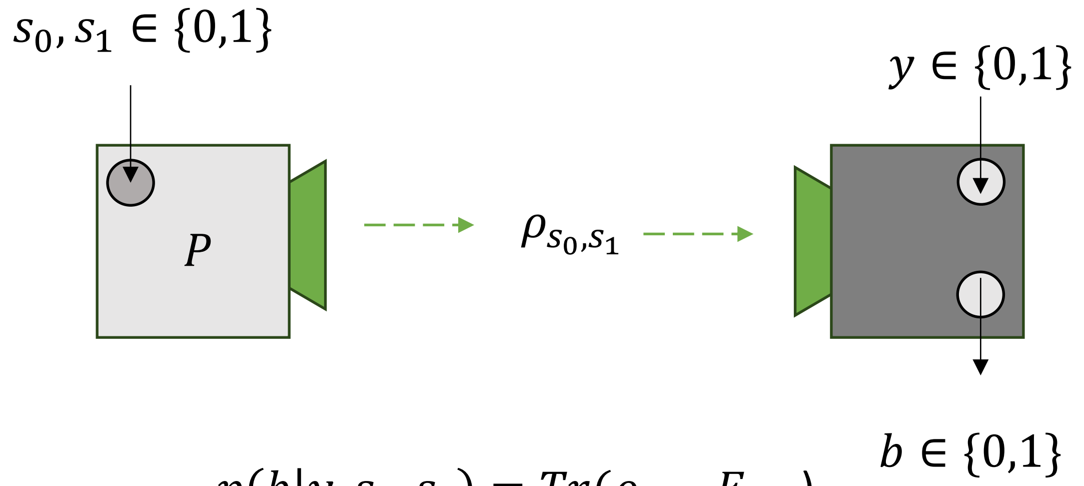

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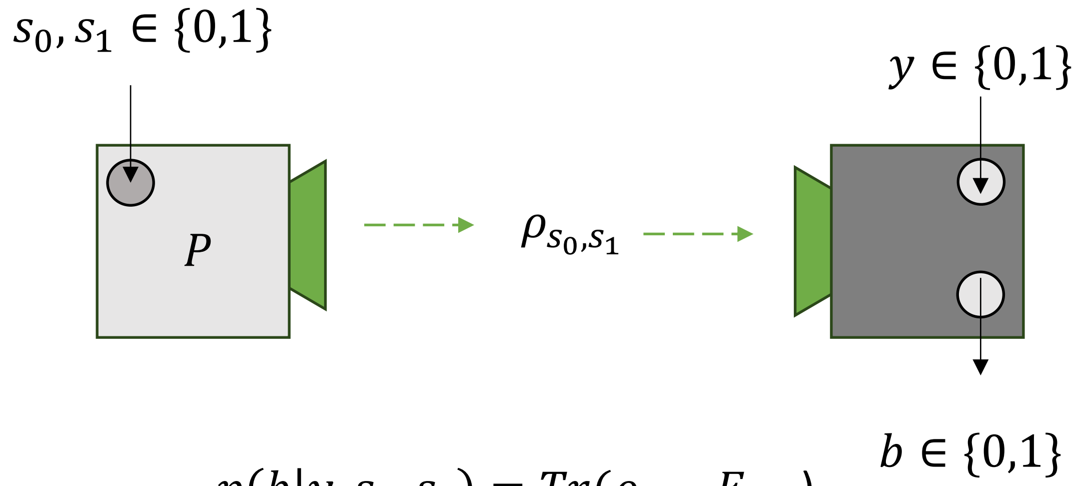

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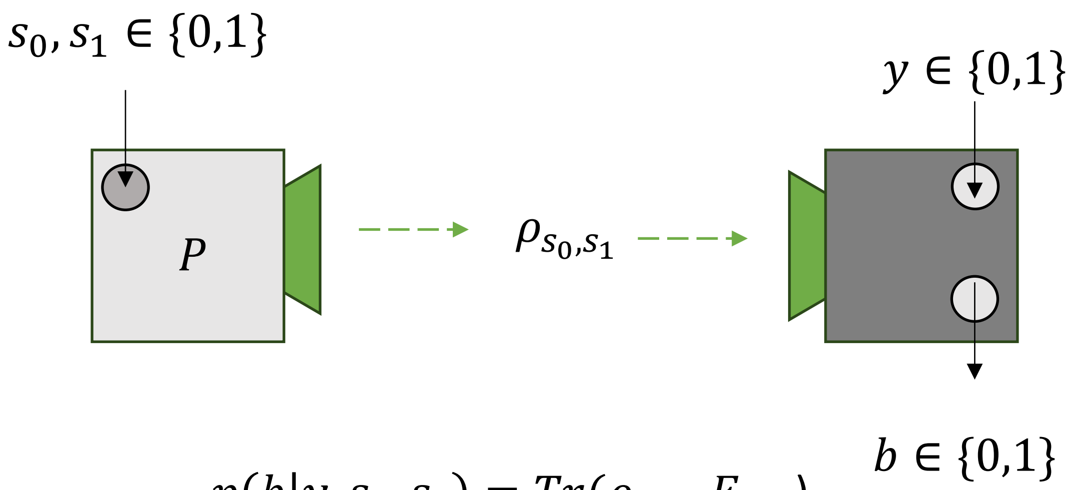

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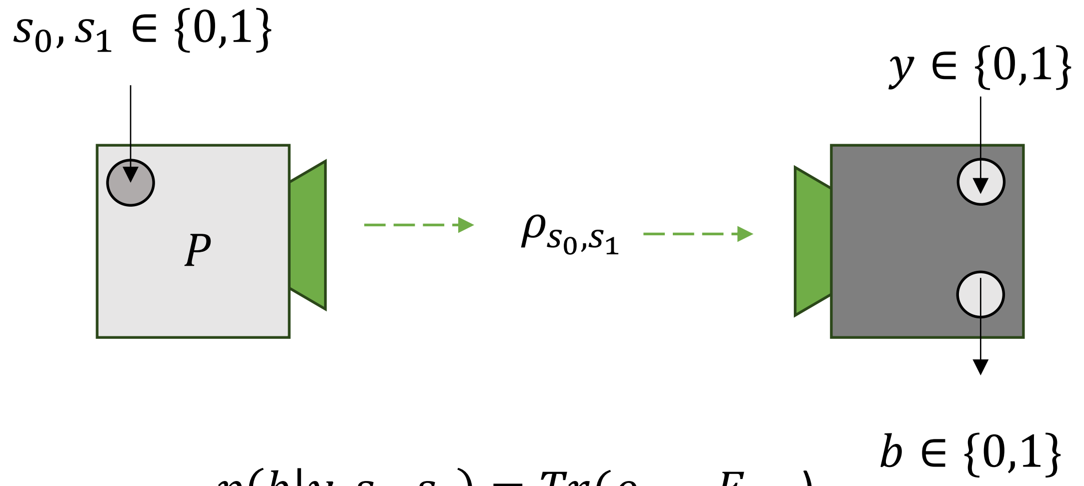

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These states form Mutually Unbiased Basis (or Conjugate Basis) i.e. Computational and Diagonal, these are states such that, when projected to the other basis no information is obtained about the state of the system.

This is the basic idea of Wiesner's Conjugate Coding paper:

A conjugate code is any communication scheme in which the physical systems used as signals are placed in states corresponding to elements of several conjugate basis of the Hilbert space describing the individual systems. Note that in the

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But even in the original conjugate coding, Wiesner already gave two applications of this idea. As we will see, under some very strong assumptions, the first example can already be rightfully claimed to be an OT.

First application of Conjugate Coding:

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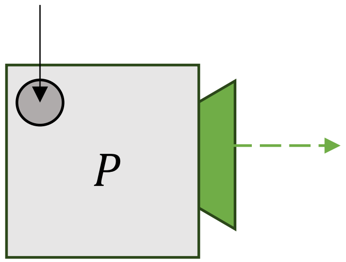
$string_{r_i}[i]$



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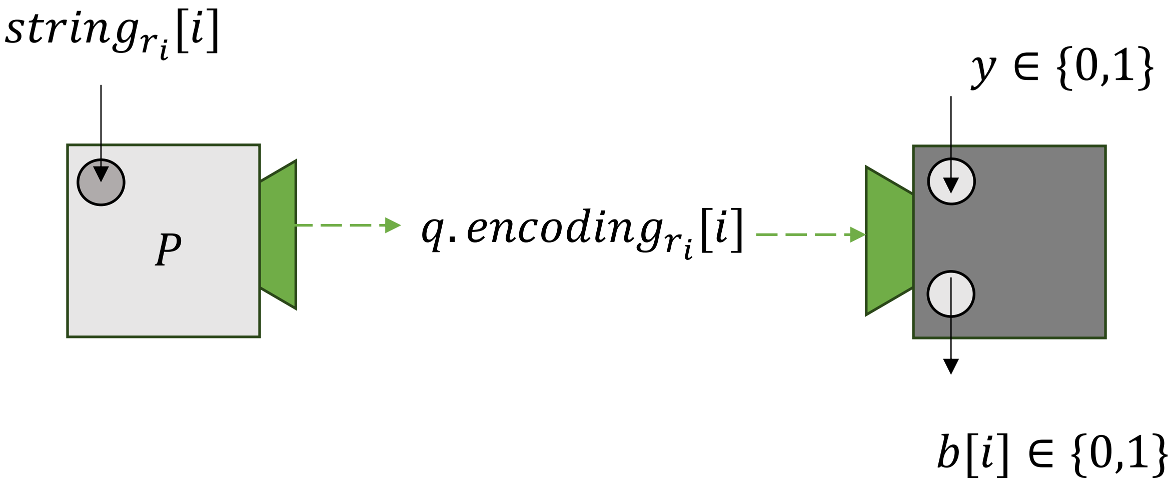


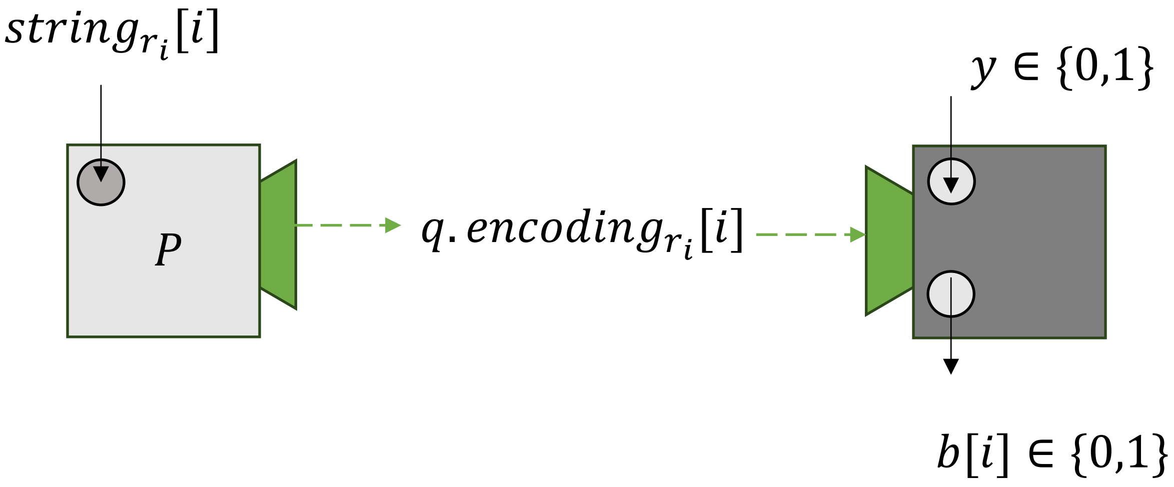
First application of Conjugate Coding:

Example One: A means for transmitting two messages either but not both of which may be received.

$string_1 = 0010100\ldots$ $\quad q.encoding_1 = |0\rangle\langle 0|, |0\rangle\langle 0|, |1\rangle\langle 1|, |0\rangle\langle 0| \ldots$

$string_2 = 1011100\ldots$ $\quad q.econding_2 = |-\rangle\langle -|, |+\rangle\langle +|, |-\rangle\langle -| \ldots$

$For\ i\ rounds, r_i \xleftarrow[\$]{} \{1,2\}$:



Bob’s measurement is not so good, so it needs to fix à priori the y globally for all rounds.

So in the end, Bob can either recover a noisy string 1 or a noisy string 2, according to his choice, but not both.
Furthermore, Alice won’t know what was the message that Bob recovered since there is no information going from Bob to Alice.

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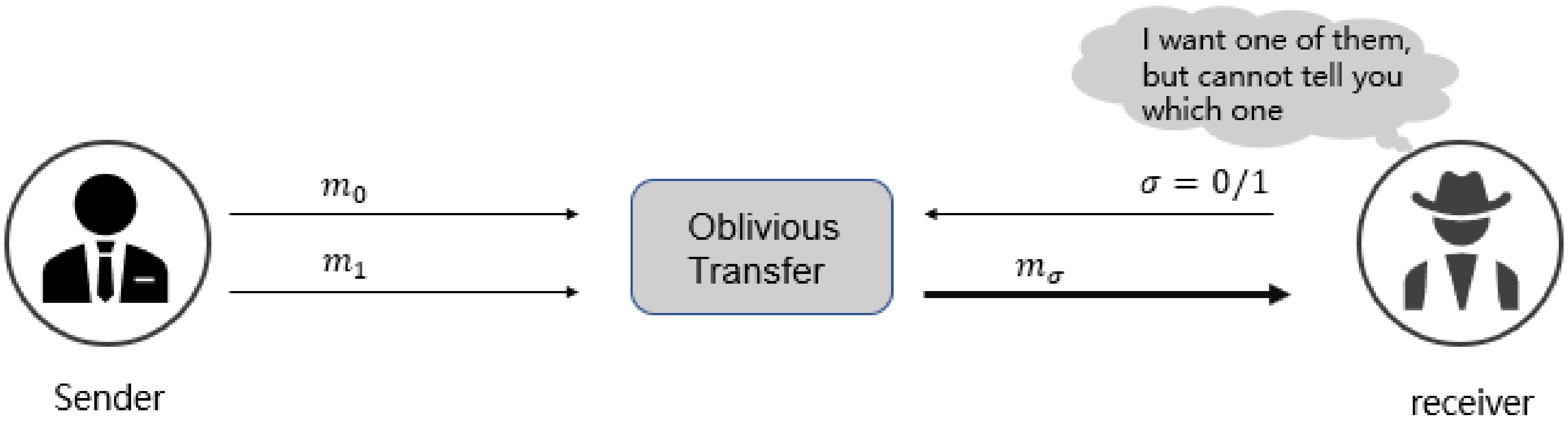

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Solved with extra physical assumptions on the trusted model, or computational assumptions.

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For the next time!

Thanks!

(You can ask me for references, I forgot to put them on the slides)